Multidimensional Heat Transfer

Heat Diffusion Equation

$$\rho c_p \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{q} = k\nabla^2 T + \dot{q}$$

- This equation governs the Cartesian, temperature distribution for a three-dimensional unsteady, heat transfer problem involving heat generation.
- For steady state $\partial / \partial t = 0$
- No generation $\dot{q} = 0$
- To solve for the full equation, it requires a total of six boundary conditions: two for each direction. Only one initial condition is needed to account for the transient behavior.

Two-D, Steady State Case

For a 2 - D, steady state situation, the heat equation is simplified to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$ it needs two boundary conditions in each direction.

There are three approaches to solve this equation:

- **Numerical Method**: Finite difference or finite element schemes, usually will be solved using computers.
- **Graphical Method**: Limited use. However, the conduction shape factor concept derived under this concept can be useful for specific configurations. (see Table 4.1 for selected configurations)
- **Analytical Method**: The mathematical equation can be solved using techniques like the method of separation of variables. (refer to handout)

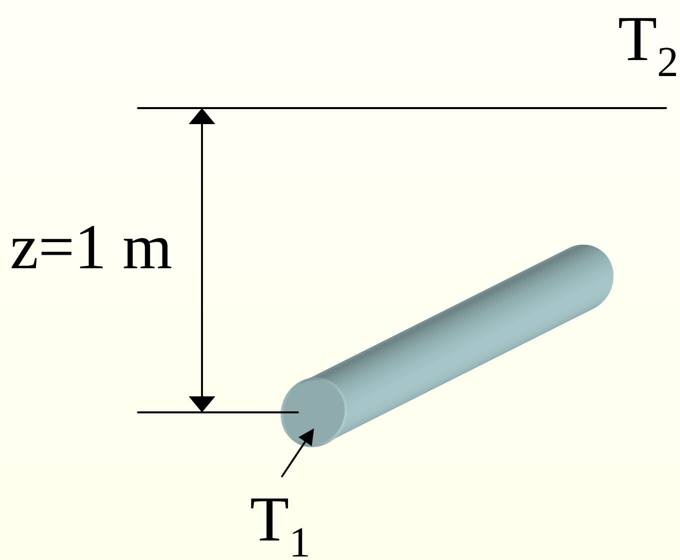
Conduction Shape Factor

This approach applied to 2-D conduction involving two isothermal surfaces, with all other surfaces being adiabatic. The heat transfer from one surface (at a temperature T_1) to the other surface (at T_2) can be expressed as: $q=Sk(T_1-T_2)$ where k is the thermal conductivity of the solid and S is the conduction shape factor.

- The shape factor can be related to the thermal resistance:
$q=Sk(T_1-T_2)=(T_1-T_2)/(1/kS)= (T_1-T_2)/R_t$
where $R_t = 1/(kS)$
- 1-D heat transfer can use shape factor also. Ex: heat transfer inside a plane wall of thickness L is $q=kA(\Delta T/L)$, $S=A/L$
- Common shape factors for selected configurations can be found in Table 4.1

Example

An Alaska oil pipe line is buried in the earth at a depth of 1 m. The horizontal pipe is a thin-walled of outside diameter of 50 cm. The pipe is very long and the averaged temperature of the oil is 100°C and the ground soil temperature is at -20 °C (k_{soil}=0.5W/m.K), estimate the heat loss per unit length of pipe.



From Table 8.7, case 1.
L>>D, z>3D/2

$$S = \frac{2\pi L}{\ln(4z/D)} = \frac{2\pi(1)}{\ln(4/0.5)} = 3.02$$

$$q = kS(T_1 - T_2) = (0.5)(3.02)(100+20)$$

$$= 181.2(W)$$ heat loss for every meter of pipe

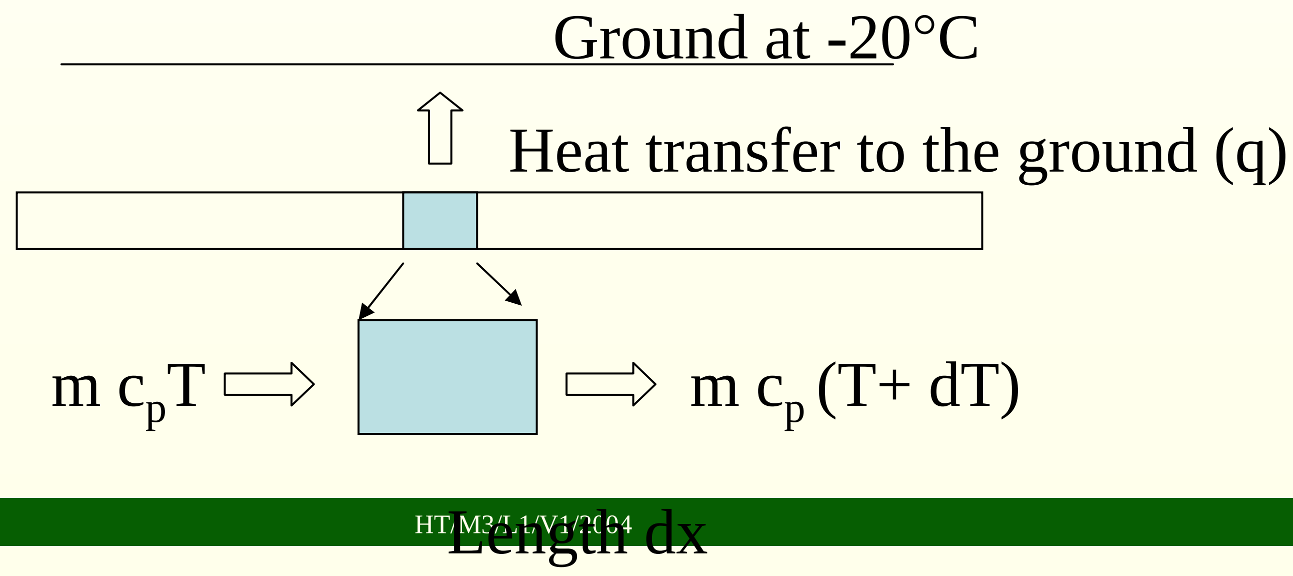
Example (cont.)

If the mass flow rate of the oil is 2 kg/s and the specific heat of the oil is 2 kJ/kg.K, determine the temperature change in 1 m of pipe length.

$$q = \dot{m}C_P\Delta T,\ \Delta T = \frac{q}{\dot{m}C_P} = \frac{181.2}{2000 * 2} = 0.045(^\circ C)$$

Therefore, the total temperature variation can be significant if the pipe is very long. For example, 45°C for every 1 km of pipe length.

- Heating might be needed to prevent the oil from freezing up.
- The heat transfer can not be considered constant for a long pipe



Example (cont.)

Heat Transfer at section with a temperature T(x)

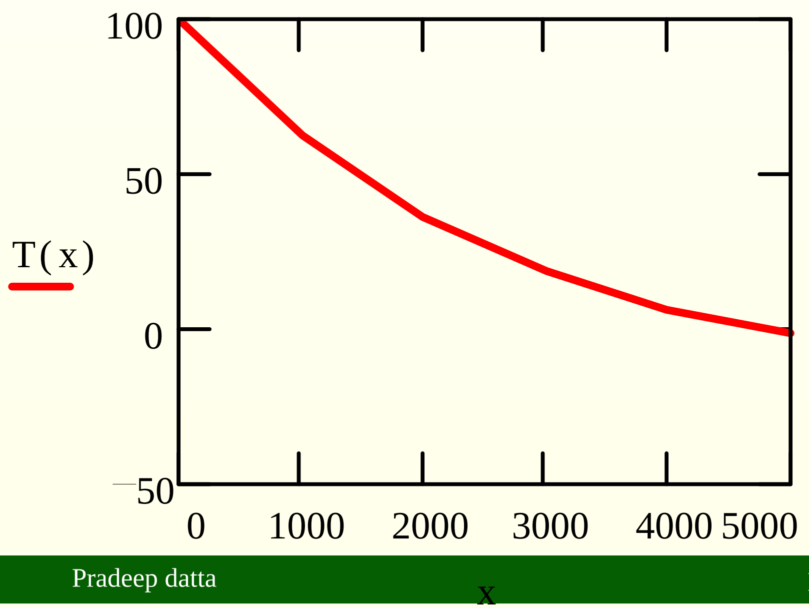
$$q = \frac{2\pi k(dx)}{\ln(4z/D)}(T+20) = 1.51(T+20)(dx)$$

Energy balance: $\dot{m}C_P T - q = \dot{m}C_P(T+dT)$

$$\dot{m}C_P\frac{dT}{dx} + 1.51(T+20) = 0,\ \frac{dT}{T+20} = -0.000378dx, \text{integrate}$$

$$T(x) = -20 + Ce^{-0.000378x},\ \text{at inlet x = 0, T(0) = 100° C, C = 120}$$

$$\text{T(x) = -20} + 120e^{-0.000378x}$$



- Temperature drops exponentially from the initial temp. of 100°C
- It reaches 0°C at x=4740 m, therefore, reheating is required every 4.7 km.

Numerical Methods

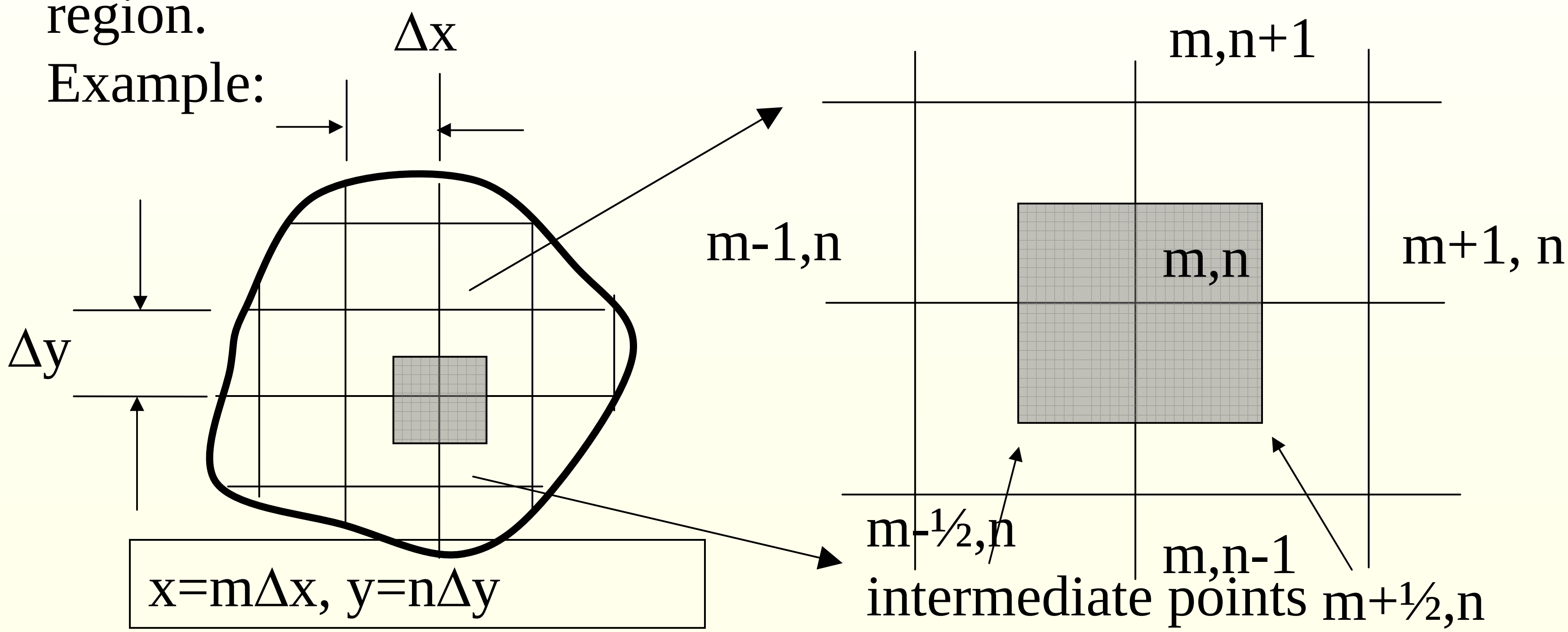
Due to the increasing complexities encountered in the development of modern technology, analytical solutions usually are not available. For these problems, numerical solutions obtained using high-speed computer are very useful, especially when the geometry of the object of interest is irregular, or the boundary conditions are nonlinear. In numerical analysis, two different approaches are commonly used: the finite difference and the finite element methods. In heat transfer problems, the finite difference method is used more often and will be discussed here. The finite difference method involves:

- Establish nodal networks
- Derive finite difference approximations for the governing equation at both interior and exterior nodal points
- Develop a system of simultaneous algebraic nodal equations
- Solve the system of equations using numerical schemes

The Nodal Networks

The basic idea is to subdivide the area of interest into sub-volumes with the distance between adjacent nodes by Dx and Dy as shown. If the distance between points is small enough, the differential equation can be approximated locally by a set of finite difference equations. Each node now represents a small region where the nodal temperature is a measure of the average temperature of the region.

Example:



Finite Difference Approximation

Heat Diffusion Equation: $\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$,

where $\alpha = \frac{k}{\rho C_P V}$ is the thermal diffusivity

No generation and steady state: $\dot{q}=0$ and $\frac{\partial}{\partial t} = 0, \Rightarrow \nabla^2 T = 0$

First, approximated the first order differentiation

at intermediate points (m+1/2,n) & (m-1/2,n)

$$\left.\frac{\partial T}{\partial x}\right|_{(m+1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m+1/2,n)} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{(m-1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m-1/2,n)} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Finite Difference Approximation (cont.)

Next, approximate the second order differentiation at m,n

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\partial T/\partial x|_{m+1/2,n} - \partial T/\partial x|_{m-1/2,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

Similarly, the approximation can be applied to the other dimension y

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Finite Difference Approximation (cont.)

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)_{m,n} \approx \frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}$$

To model the steady state, no generation heat equation: $\nabla^2 T = 0$

This approximation can be simplified by specify $\Delta x = \Delta y$

and the nodal equation can be obtained as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

This equation approximates the nodal temperature distribution based on the heat equation. This approximation is improved when the distance between the adjacent nodal points is decreased:

Since $\lim(\Delta x \to 0)\frac{\Delta T}{\Delta x}=\frac{\partial T}{\partial x}, \lim(\Delta y \to 0)\frac{\Delta T}{\Delta y}=\frac{\partial T}{\partial y}$

A System of Algebraic Equations

- The nodal equations derived previously are valid for all interior points satisfying the steady state, no generation heat equation. For each node, there is one such equation.
For example: for nodal point m=3, n=4, the equation is
$T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5} - 4T_{3,4} = 0$
$T_{3,4} = (1/4)(T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5})$

- Nodal relation table for exterior nodes (boundary conditions) can be found in standard heat transfer textbooks (Table 4.2 of our textbook).

- Derive one equation for each nodal point (including both interior and exterior points) in the system of interest. The result is a system of N algebraic equations for a total of N nodal points.

Matrix Form

The system of equations:

$$
\begin{aligned}
a_{11}T_1 + a_{12}T_2 + \cdots + a_{1N}T_N &= C_1 \\
a_{21}T_1 + a_{22}T_2 + \cdots + a_{2N}T_N &= C_2 \\
\vdots \qquad \vdots \qquad \vdots \quad \vdots \qquad & \quad \vdots \\
a_{N1}T_1 + a_{N2}T_2 + \cdots + a_{NN}T_N &= C_N
\end{aligned}
$$

A total of N algebraic equations for the N nodal points and the system can be expressed as a matrix formulation: [**A**][**T**]=[**C**]

$$
where\ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}
$$

Numerical Solutions

Matrix form: [**A**][**T**]=[**C**].
From linear algebra: $[\mathbf{A}]^{-1}[\mathbf{A}][\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$, $[\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$
where $[\mathbf{A}]^{-1}$ is the inverse of matrix [**A**]. [**T**] is the solution vector.

• Matrix inversion requires cumbersome numerical computations and is not efficient if the order of the matrix is high (>10)

• Gauss elimination method and other matrix solvers are usually available in many numerical solution package. For example, "Numerical Recipes" by Cambridge University Press or their web source at www.nr.com.

• For high order matrix, iterative methods are usually more efficient. The famous Jacobi & Gauss-Seidel iteration methods will be introduced in the following.

Iteration

General algebraic equation for nodal point:

$$\sum_{j=1}^{i-1} a_{ij} T_j + a_{ii} T_i + \sum_{j=i+1}^{N} a_{ij} T_j = C_i,$$

$$(\text{Example}: a_{31}T_1 + a_{32}T_2 + a_{33}T_3 + \cdots + a_{1N}T_N = C_1,\ i = 3)$$

Rewrite the equation of the form:

Replace (k) by (k-1) for the Jacobi iteration

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} T_j^{(k)} - \sum_{j=i+1}^{N} \frac{a_{ij}}{a_{ii}} T_j^{(k-1)}$$

- (k) - specify the level of the iteration, (k-1) means the present level and (k) represents the new level.
- An initial guess (k=0) is needed to start the iteration.
- By substituting iterated values at (k-1) into the equation, the new values at iteration (k) can be estimated
- The iteration will be stopped when $\max\left| T_i^{(k)} - T_i^{(k-1)} \right| \le \varepsilon$, where ε specifies a predetermined value of acceptable error

Example

Solve the following system of equations using (a) the Jacobi methos, (b) the Gauss Seidel iteration method.

$$\begin{bmatrix} 4 & 2 & 1 \\ -1 & 2 & 0 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \\ 16 \end{bmatrix}$$

$$4X + 2Y + Z = 11,$$

$$-X + 2Y + 0 * Z = 3,$$

$$2X + Y + 4Z = 16$$

Reorganize into new form:

$$X = \frac{11}{4} - \frac{1}{2}Y - \frac{1}{4}Z$$

$$Y = \frac{3}{2} + \frac{1}{2}X + 0 * Z$$

$$Z = 4 - \frac{1}{2}X - \frac{1}{4}Y$$

(a) Jacobi method: use initial guess $X^0=Y^0=Z^0=1$,

stop when $\max\left| X^k-X^{k-1}, Y^k-Y^{k-1}, Z^k-Z^{k-1} \right| \le 0.1$

First iteration:

$X^1= (11/4) - (1/2)Y^0 - (1/4)Z^0 = 2$

$Y^1= (3/2) + (1/2)X^0 = 2$

$Z^1= 4 - (1/2) X^0 - (1/4)Y^0 = 13/4$

Example (cont.)

Second iteration: use the iterated values $X^1=2$, $Y^1=2$, $Z^1=13/4$

$X^2 = (11/4) - (1/2)Y^1 - (1/4)Z^1 = 15/16$

$Y^2 = (3/2) + (1/2)X^1 = 5/2$

$Z^2 = 4 - (1/2) X^1 - (1/4)Y^1 = 5/2$

Converging Process:

$$[1,1,1], \left[2,2,\frac{13}{4}\right], \left[\frac{15}{16},\frac{5}{2},\frac{5}{2}\right], \left[\frac{7}{8},\frac{63}{32},\frac{93}{32}\right], \left[\frac{133}{128},\frac{31}{16},\frac{393}{128}\right]$$

$$\left[\frac{519}{512},\frac{517}{256},\frac{767}{256}\right]$$. Stop the iteration when

$$\max \left|X^5 - X^4, Y^5 - Y^4, Z^5 - Z^4\right| \leq 0.1$$

Final solution [1.014, 2.02, 2.996]

Exact solution [1, 2, 3]

Example (cont.)

(b) Gauss-Seidel iteration: Substitute the iterated values into the iterative process immediately after they are computed.

Use initial guess $X^0 = Y^0 = Z^0 = 1$

$$X = \frac{11}{4} - \frac{1}{2}Y - \frac{1}{4}Z, \; Y = \frac{3}{2} + \frac{1}{2}X, \; Z = 4 - \frac{1}{2}X - \frac{1}{4}Y$$

First iteration: $X^1 = \frac{11}{4} - \frac{1}{2}(Y^0) - \frac{1}{4}(Z^0) = 2$

Immediate substitution

$$Y^1 = \frac{3}{2} + \frac{1}{2}X^1 = \frac{3}{2} + \frac{1}{2}(2) = \frac{5}{2}$$

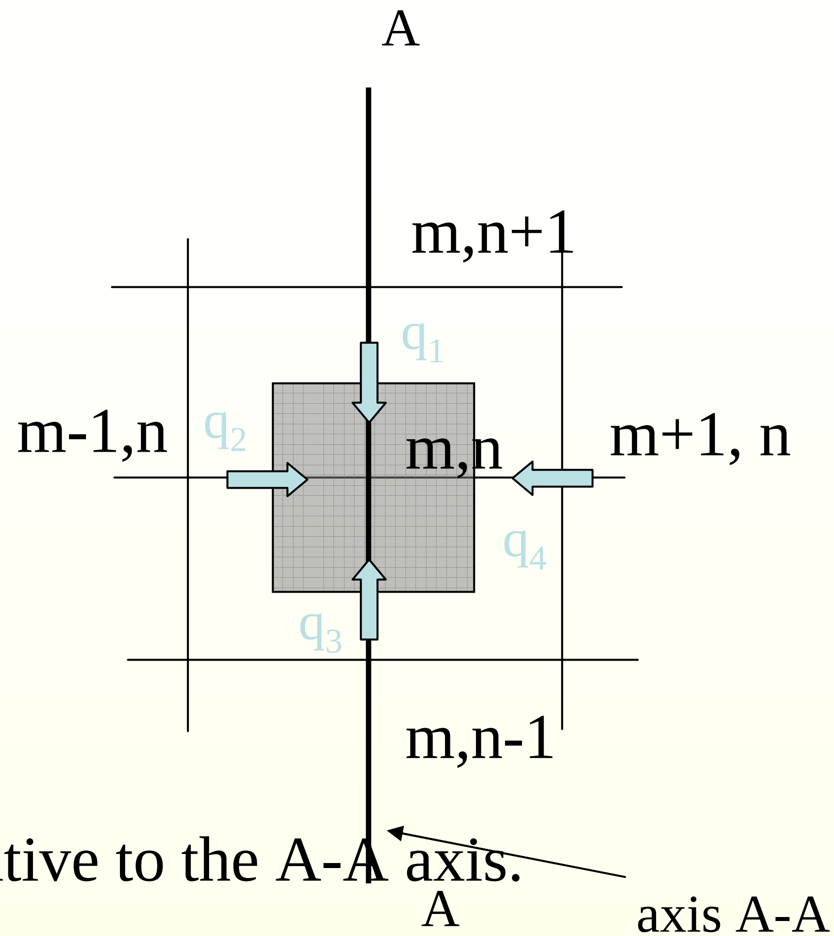
$$Z^1 = 4 - \frac{1}{2}X^1 - \frac{1}{4}Y^1 = 4 - \frac{1}{2}(2) - \frac{1}{4}\left(\frac{5}{2}\right) = \frac{19}{8}$$

Converging process: $[1,1,1], \left[2, \frac{5}{2}, \frac{19}{8}\right], \left[\frac{29}{32}, \frac{125}{64}, \frac{783}{256}\right], \left[\frac{1033}{1024}, \frac{4095}{2048}, \frac{24541}{8192}\right]$

The iterated solution [1.009, 1.9995, 2.996] and it converges faster

Numerical Method (Special Cases)

For all the special cases discussed in the following, the derivation will be based on the standard nodal point configuration as shown to the right.



- Symmetric case: symmetrical relative to the A-A axis.

In this case, $T_{m-1,n}=T_{m+1,n}$

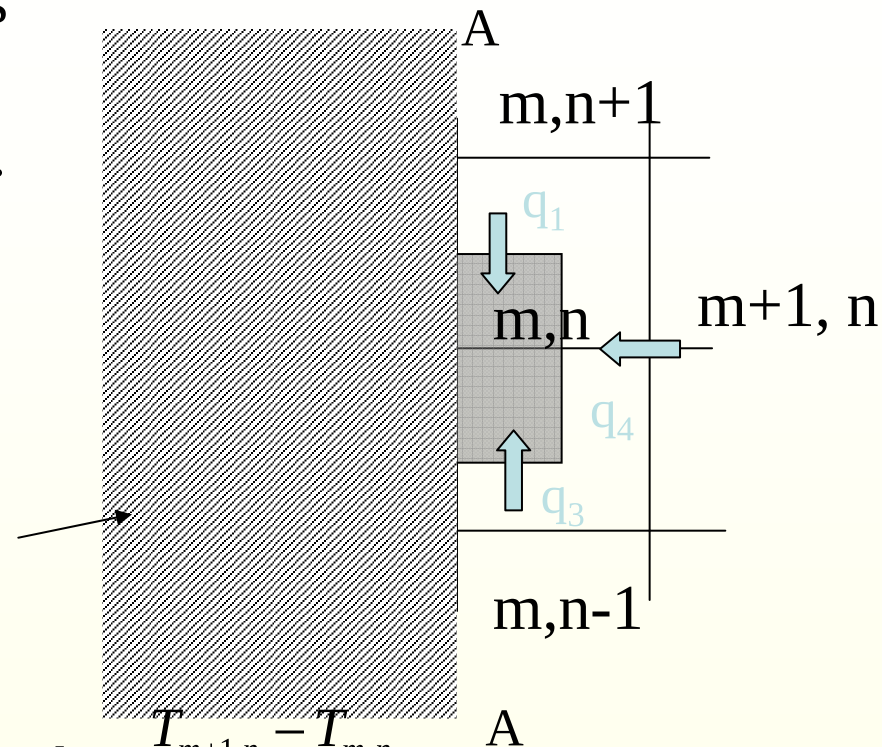
Therefore the standard nodal equation can be written as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}$$

$$=2T_{m+1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

Special cases (cont.)

- Insulated surface case: If the axis A-A is an insulated wall, therefore there is no heat transfer across A-A. Also, the surface area for q_1 and q_3 is only half of their original value. Write the energy balance equation ($q_2=0$):



Insulated surface

$$q_1 + q_3 + q_4 = 0$$

$$k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n+1}-T_{m,n}}{\Delta y} + k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1}-T_{m,n}}{\Delta y} + k\Delta y\frac{T_{m+1,n}-T_{m,n}}{\Delta x} = 0$$

$$2T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0$$

This equation is identical to the symmetrical case discussed previously.

Special cases (cont.)

□ With internal generation G=gV where g is the power generated per unit volume (W/m^3). Based on the energy balance concept:

$$q_1 + q_2 + q_3 + q_4 + G$$

$$q_1 + q_2 + q_3 + q_4 + g(\Delta x)(\Delta y)(1) = 0$$

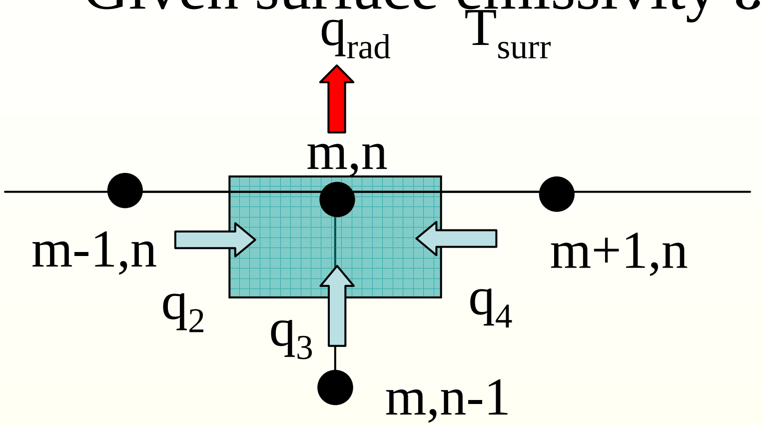
Use 1 to represent the dimension along the z-direction.

$$k(T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n}) + g\left(\Delta x\right)^2 = 0$$

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{g(\Delta x)^2}{k} = 0$$

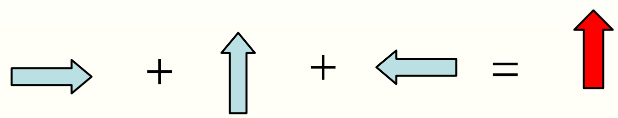
Special cases (cont.)

- Radiation heat exchange with respect to the surrounding (assume no convection, no generation to simplify the derivation). Given surface emissivity ε, surrounding temperature T_{surr}.



From energy balance concept:

$q_2+q_3+q_4=q_{rad}$



$$k\left(\frac{\Delta y}{2}\right)\frac{T_{m-1,n}-T_{m,n}}{\Delta x}+k\left(\Delta x\right)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}+k\left(\frac{\Delta y}{2}\right)\frac{T_{m+1,n}-T_{m,n}}{\Delta x}=\varepsilon\sigma\left(\Delta x\right)\left(T_{m,n}{}^{4}-T_{surr}{}^{4}\right)$$

$$k\left(T_{m-1,n}+T_{m+1,n}+2T_{m,n-1}-4T_{m,n}\right)=2\varepsilon\sigma\left(\Delta x\right)\left(T_{m,n}^{4}-T_{surr}^{4}\right)$$

$$T_{m-1,n}+T_{m+1,n}+2T_{m,n-1}-4T_{m,n}-\frac{2\varepsilon\sigma\left(\Delta x\right)}{k}T_{m,n}^{4}=-2\frac{\varepsilon\sigma\left(\Delta x\right)}{k}T_{surr}^{4}$$

Non-linear term, can solve using the iteration method